NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 1

2024 Year 12 Course Assessment Task 4 (Trial Examination)
Wednesday 14 August, 2024

General instructions

- Working time – 2 hours.
  (plus 10 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer grid provided (on page 13)

SECTION II

- Commence each new question on a new booklet. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

NESA STUDENT #: .... **# BOOKLETS USED:** .....

Class (please ✔)

- ◯ 12MXX.1 – Ms Ham
- ◯ 12MXX.2 – Mr Ho
- ◯ 12MXX.3 – Mr Lam
- ◯ 12MAX.1 – Ms J Kim
- ◯ 12MAX.2 – Mrs Bhamra
- ◯ 12MAX.3 – Ms Lee

Marker's use only.

QUESTION	1-10	11	12	13	14	Total	%
MARKS	/10	/16	/12	/17	/15	/70	

Section I

10 marks
Attempt Question 1 to 10
Allow approximately 15 minutes for this section

Mark your answers on the answer grid provided (labelled as page 13).

Questions **Marks**

1. Which of the following is the angle between the vectors $\underset{\sim}{p} = -\underset{\sim}{i} + 4\underset{\sim}{j}$ and $\underset{\sim}{q} = 7\underset{\sim}{i} - 3\underset{\sim}{j}$, correct to the nearest degree? **1**

(A) 53°

(B) 81°

(C) 109°

(D) 127°

2. Which of the following is the integral of $\displaystyle\int \sin 2x \cos 6x \, dx$? **1**

(A) $-\dfrac{1}{16}\cos 8x + \dfrac{1}{8}\cos 4x + C$

(B) $\dfrac{1}{16}\cos 8x - \dfrac{1}{8}\cos 4x + C$

(C) $-\dfrac{1}{4}\cos 2x + \dfrac{1}{8}\cos 4x + C$

(D) $\dfrac{1}{4}\cos 2x - \dfrac{1}{8}\cos 4x + C$

3. Which of the following is a particular solution to the differential equation? **1**

$$2y - 3\frac{dy}{dx} = 1$$

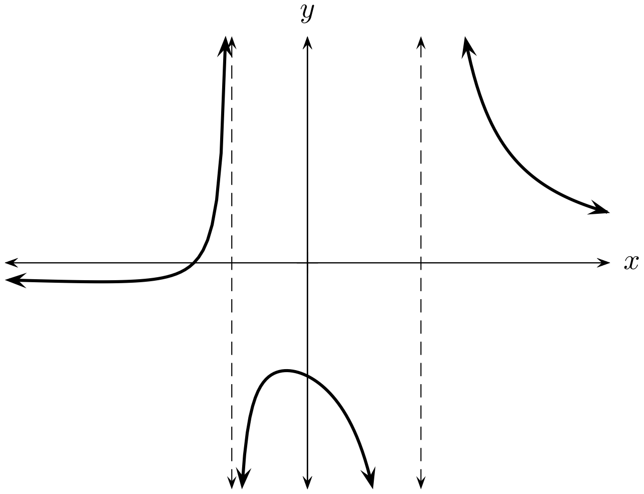
(A) $y = -\dfrac{1}{3}$

(B) $y = -e^{\frac{1}{3}(2x-1)}$

(C) $y = \dfrac{1}{2}$

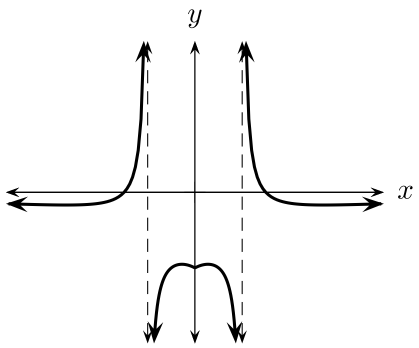
(D) $y = 3e^{\frac{2}{3}x}$

4. The graph of function $y = f(x)$ is shown below. **1**

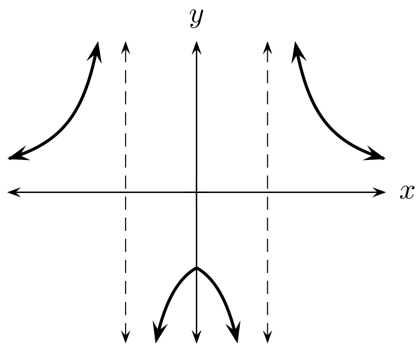


Which of the following is the graph of $|y| = f(x)$?

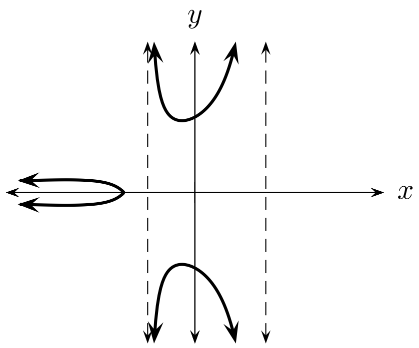
(A)



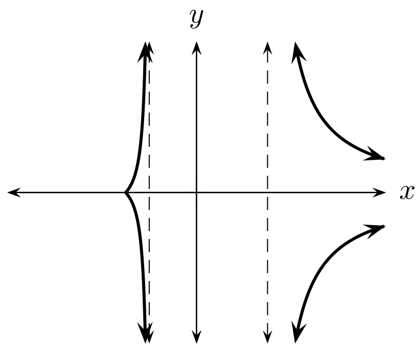
(C)



(B)



(D)



Examination continues overleaf...

5. In how many ways can all the letters of the word `REGRESSION` be placed in a line with the two `R`s together? **1**

(A) 90 720

(B) 181 440

(C) 362 880

(D) 907 200

6. Ignoring the constant of integration, which of the following is the primitive of **1**

$$\frac{1}{1+3(x-2)^2}$$

(A) $\sqrt{3}\tan^{-1}\left(\sqrt{3}(x-2)\right)$

(B) $\frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}(x-2)\right)$

(C) $\sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}(x-2)\right)$

(D) $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}(x-2)\right)$

7. Which of the following is the derivative of **1**

$$\cos^{-1}\left(\sin^{-1}x\right)$$

(A) $\sqrt{\frac{1-x^2}{1-(\sin^{-1}x)^2}}$

(B) $\frac{1}{\sqrt{\left(1-(\sin^{-1}x)^2\right)(1-x^2)}}$

(C) $\frac{-\sin^{-1}x}{\sqrt{1-(\sin^{-1}x)^2}}$

(D) $\frac{-1}{\sqrt{\left(1-(\sin^{-1}x)^2\right)(1-x^2)}}$

8. The function $f(x)$ has parametric equations: **1**

$$x = e^{-t} \quad , \quad y = 3t^2 - t^3$$

Which of the following is the derivative $\frac{dy}{dx}$?

(A) $-e^{-t}(6t-3t^2)$

(B) $\frac{-e^{-t}}{6t-3t^2}$

(C) $-e^{t}(6t-3t^2)$

(D) $\frac{-e^{t}}{6t-3t^2}$

9. Mathematical induction is used to prove for all integers $n \geq 1$. **1**

$$\left(1 \times 1! + \frac{0}{1!}\right) + \left(2 \times 2! + \frac{1}{2!}\right) + \ \cdots \ + \left(n \times n! + \frac{n-1}{n!}\right) = (n+1)! - \frac{1}{n!}$$

Which of the following is a correct expression for part of the inductive step in the proof? (i.e. when proving the statement is true for $n = k + 1$)

(A) $\text{LHS} = (k+1)! + \dfrac{k}{(k+1)!} - \dfrac{1}{k!}$

(B) $\text{LHS} = (k+2)! + \dfrac{k}{(k+1)!} - \dfrac{1}{k!}$

(C) $\text{LHS} = (k+1)! + k \times k! + \dfrac{k-1}{k!}$

(D) $\text{LHS} = (k+2)! + \dfrac{k-1}{k!} - \dfrac{1}{(k-1)!}$

10. Let $\underset{\sim}{a} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} -1 \\ k \end{pmatrix}$, where $-2 \leq k \leq 0$. **1**

Which of the following vectors is a possible projection of $\underset{\sim}{b}$ on to $\underset{\sim}{a}$?

(A) $\dfrac{2}{5}\hat{\underset{\sim}{a}}$ (B) $\dfrac{9}{5}\hat{\underset{\sim}{a}}$ (C) $\dfrac{2}{5}\hat{\underset{\sim}{b}}$ (D) $\dfrac{9}{5}\hat{\underset{\sim}{b}}$

Examination continues overleaf…

Section II

60 marks
Attempt Questions 11 to 14
Allow approximately 1 hour and 45 minutes for this section.

Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 Marks) Commence a NEW booklet. **Marks**

(a) Solve the inequality for x **3**

$$\frac{(2x+1)^2}{x-1} \le -3$$

(b) Consider the function $f(x) = 6\sin x - 8\cos x$.

i. Find R and α such that $6\sin x - 8\cos x \equiv R\sin(x-\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give your value of α correct to two decimal places. **2**

ii. Hence find the minimum value of $f(x)$ and the smallest positive value of x for which this minimum occurs. Give your answer correct to two decimal places. **2**

(c) Consider the function $f(x) = \sin^{-1}\left(\frac{1}{6}x + \frac{1}{2}\right)$.

i. State the domain and range of $f(x)$. **2**

ii. Sketch the graph of $f(x)$, clearly indicating the coordinates of the x and y-intercepts and the endpoints of the graph. **2**

iii. Hence sketch the graph of $y = \frac{1}{f(x)}$, clearly indicating the coordinates of any intercepts and the endpoints of the graph. **2**

(d) Prove by mathematical induction that $7^{2k+1} + 2^{2k+1}$ is divisible by 9 for all integers $k \ge 1$. **3**

Question 12 (12 Marks) Commence a NEW booklet. **Marks**

(a) Using the substitution $t = \tan\dfrac{x}{2}$, solve the following equation for $0 \le x \le 2\pi$: **3**

$$2 - 3\sin x + 2\cos x = 0$$

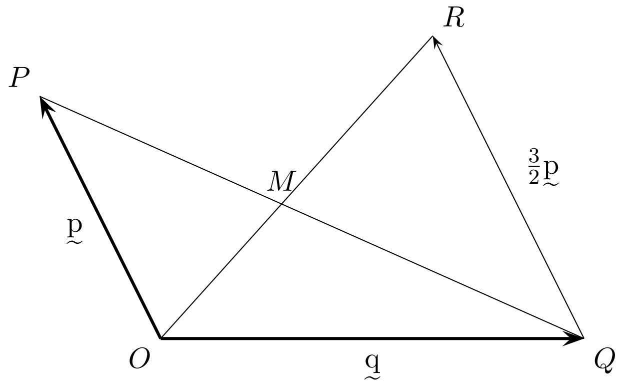
Round your answer(s) correct to two decimal places where necessary.

(b) i. How many ways can six people be seated at circular table if two of them must sit next to each other? **1**

ii. A group of eight coworkers are to be seated at two circular tables: six on one table, and two on the other table. Three of the coworkers, Andy, Lewis and Kelly are on bad terms, so no two of these three can sit next to each other. **3**

Using part (i) or otherwise, find how many ways the eight coworkers can be seated at these two tables such that Andy, Lewis and Kelly are separated in this way?

(c) In the diagram, the position vectors of P and Q respectively are $\underset{\sim}{p}$ and $\underset{\sim}{q}$, and M is a point on PQ. It is given that $\overrightarrow{QR} = \dfrac{3}{2}\overrightarrow{OP}$. The ratios $PM : MQ = \lambda : 1$ and $OM : MR = \mu : 1$ are also given.



i. Show that $\overrightarrow{OM} = \dfrac{\mu}{\mu+1}\left(\dfrac{3}{2}\underset{\sim}{p} + \underset{\sim}{q}\right)$. **1**

ii. Show that $\overrightarrow{OM} = \underset{\sim}{p} + \dfrac{\lambda}{\lambda+1}\left(\underset{\sim}{q} - \underset{\sim}{p}\right)$. **1**

iii. Prove that $\lambda = \dfrac{2}{3}$, and hence show that $\overrightarrow{OM} = \dfrac{3}{5}\underset{\sim}{p} + \dfrac{2}{5}\underset{\sim}{q}$. **3**

Examination continues overleaf...

Question 13 (17 Marks) Commence a NEW booklet. **Marks**

(a) The polynomial $P(x) = ax^3 + bx^2 + cx + d$ has roots $x = -\dfrac{1}{2}$ and $x = \gamma$.

i. Given that $x = -\dfrac{1}{2}$ is a double root, show that **3**

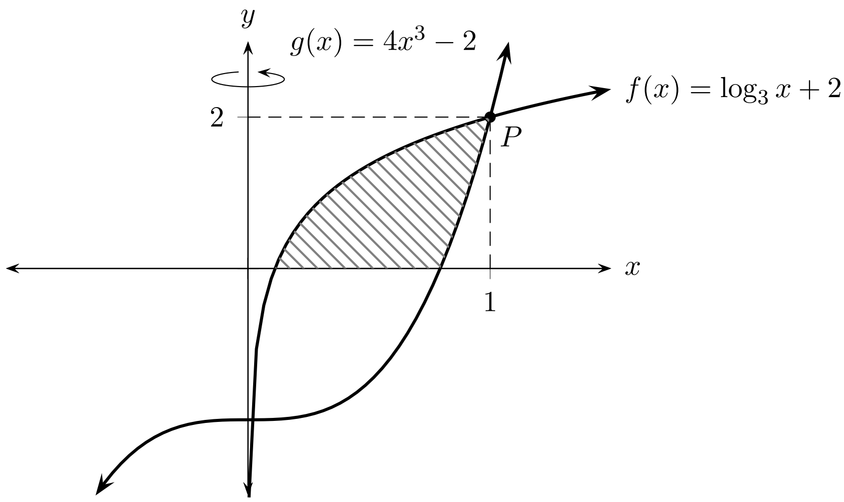
$$4\left(\frac{c}{a}\right) - 16\left(\frac{d}{a}\right) = 1$$

ii. Hence or otherwise, show that if $3c = 11d$, then $\gamma = 3$. **2**

(b) Use the substitution $x = \cos\theta$, where $0 \le \theta \le \frac{\pi}{2}$, to evaluate **4**

$$\int_0^{\frac{1}{2}} \sqrt{1 - x^2}\, dx$$

(c) The graphs of $f(x) = \log_3 x + 2$ and $g(x) = 4x^3 - 2$ are known to intersect at $P(1, 2)$, as shown in the diagram below. **4**



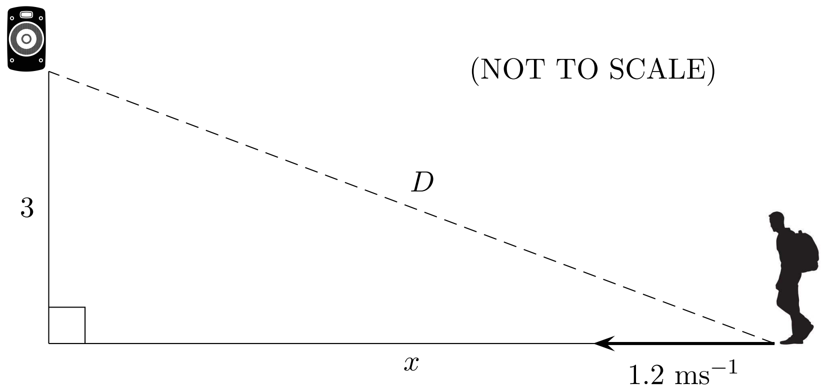
Calculate the volume of the solid of revolution formed between $f(x)$, $g(x)$ and the x-axis when the region is rotated about the y-axis. Give your answer correct to two decimal places.

(d) Sound intensity I in watts per square metre (W/m^2) is inversely proportional to D^2, the squared distance directly between the source and the listener in metres: **4**

$$I = \frac{P}{4\pi D^2}$$

where P is a fixed constant that represents the sound power in watts (W).

A student is walking down the corridor at 1.2 ms^{-1} towards a speaker mounted to the ceiling, which is 3 metres above the floor.



The speaker plays a chime with a sound power of 10 W.

Find the rate of change in sound intensity of the chime when the direct distance between the student and the speaker is 8 metres. Give your answer correct to three decimal places.

Examination continues overleaf...

Question 14 (15 Marks) Commence a NEW booklet. **Marks**

(a) Consider the binomial expression

$$\left(3x^a - \frac{1}{x^{4a}}\right)^{3n}$$

where a and n are positive integers.

i. Show that the expression will only have a term independent of x if n is multiple of 5. **3**

ii. Hence, or otherwise, find the term independent of x in $\left(3x^2 - \frac{1}{x^8}\right)^{75}$. **2**

(b) On a certain island, a researcher released 400 mosquitoes. The population growth is given by the logistic equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{16\,000}\right)$$

where P is the population of mosquitoes, t is the number of years after the mosquitoes were released, and r is a constant.

Five years after the mosquitoes were released, the population of mosquitoes was estimated to be 4600.

It is given that $\dfrac{1}{P(16\,000 - P)} = \dfrac{1}{16\,000}\left(\dfrac{1}{P} + \dfrac{1}{16\,000 - P}\right)$.

i. Show by integration that $P = \dfrac{16\,000}{39e^{-rt} + 1}$. **4**

ii. Hence show that $r = -\dfrac{1}{5}\ln\left(\dfrac{19}{299}\right)$. **1**

iii. When is the population of mosquitoes growing at its fastest rate? **2**

iv. Find the island's carrying capacity for the population of mosquitoes. **1**

The researcher determines that if the population of mosquitoes exceeds 13 000 it will harm the ecosystem of the island. Spiders are introduced to the island to control the mosquito population, reducing it by 9% every year. The modified logistic equation for the population growth is

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{16\,000}\right) - 0.09P$$

v. Determine if the researcher will be able to protect the ecosystem from harm. Justify your answer with the appropriate calculations. **2**

End of paper.

Sample Band E4 Responses

Section I

1. (D) **2.** (A) **3.** (C) **4.** (D) **5.** (A)
6. (B) **7.** (D) **8.** (C) **9.** (B) **10.** (B)

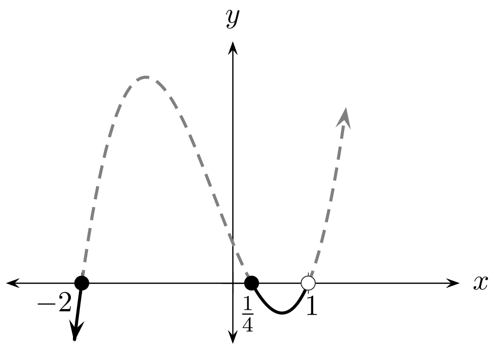
Section II

Question 11

(a) (3 marks) - (Lam)

- ✓ [1] for significant progress in simplifying the inequality. **Note:** terminating error if not multiplying by the square of the denominator (or the equivalent).
- ✓ [1] for final solution with one error
- ✓ [1] for final solution noting that $x \neq 1$

$$\begin{aligned}
\frac{(2x+1)^2}{x-1} &\le -3 \\
(x-1)(2x+1)^2 &\le -3(x-1)^2 \\
(x-1)\left[(2x+1)^2 + 3(x-1)\right] &\le 0 \\
(x-1)(4x^2+7x-2) &\le 0 \\
(x-1)(4x-1)(x+2) &\le 0 \qquad \checkmark
\end{aligned}$$



From the graph (noting that $x \neq 1$):

$$x \le -2 \text{ or } -\frac{1}{4} \le x < 1 \qquad \checkmark\checkmark$$

(b) i. (2 marks) - (Lam)

✓ [1] for correct value of R with sufficient working

✓ [1] for correct value of α with sufficient working

$$\begin{aligned} \text{Let} \quad 6\sin x - 8\cos x &\equiv R\sin(x-\alpha) \\ &= R\cos\alpha\sin x - R\sin\alpha\cos x \end{aligned}$$

By equating coefficients:

$$R\cos\alpha = 6 \qquad (1)$$
$$-R\sin\alpha = -8 \qquad (2)$$

Equation $(1)^2 + (2)^2$:

$$R^2 = 100$$
$$\therefore R = 10 \quad ✓$$

Equation $(2) \div (1)$:

$$\tan\alpha = \frac{4}{3}$$
$$\therefore \alpha = 0.93 \quad ✓ \qquad \text{(correct to two d.p.)}$$

ii. (2 marks) - (Lam)

✓ [1] for correct minimum value

✓ [1] for correct value of x

$$f(x) = 10\sin(x - 0.93) \qquad \text{(from i)}$$

$\therefore$ Minimum value of $f(x) = -10$. ✓

$f(x)$ is a sine graph translated 0.93 units to the right:

$$x = \frac{3\pi}{2} + 0.93$$
$$\therefore x = 5.64 \quad ✓ \qquad \text{(correct to two d.p.)}$$

(c) i. (2 marks) - (Lam)

✓ [1] for correct domain

✓ [1] for correct range

Domain of $f(x)$:

$$\begin{aligned} -1 \leq \frac{1}{6}x + \frac{1}{2} &\leq 1 \\ -\frac{3}{2} \leq \frac{1}{6}x &\leq \frac{1}{2} \\ -9 \leq x &\leq 3 \quad \checkmark \end{aligned}$$

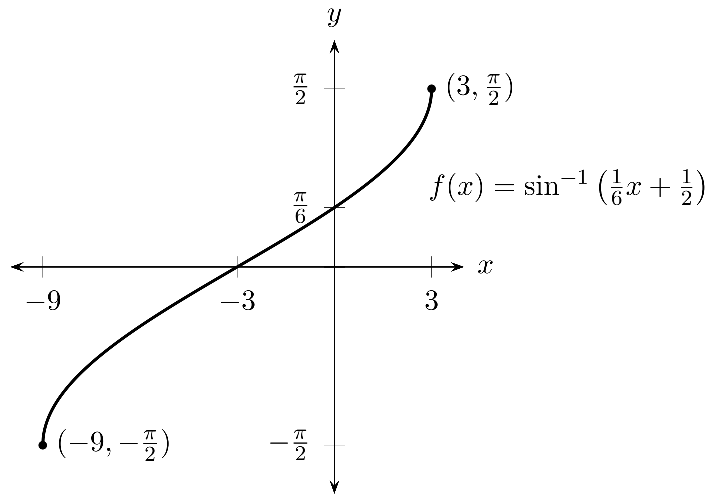
Range of $f(x)$:

$$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \quad \checkmark$$

ii. (2 marks)

✓ [1] for correct sine inverse graph shape

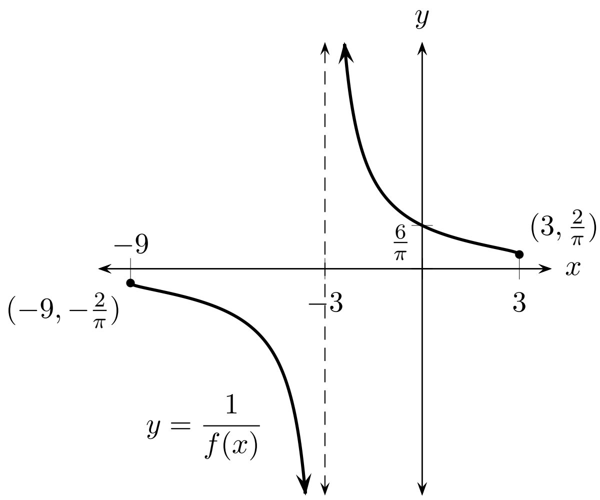
✓ [1] for correct graph with scale, axes, intercepts and endpoints clearly labelled



iii. (2 marks) - (Lam)

✓ [1] for correct shape of graph

✓ [1] for correct graph with scale, axes, intercepts and endpoints clearly labelled



(d) (3 marks) - (Bhamra)

- ✓ [1] for proving the statement when $k = 1$
- ✓ [1] for correctly applying the assumption
- ✓ [1] for correct final proof

When $k = 1$:

$$\begin{aligned} 7^3 + 2^3 &= 351 \\ &= 9 \times 39 \end{aligned}$$

$\therefore$ Statement is true when $k = 1$. ✓

Assume the statement is true when $k = r$:

$$\begin{aligned} 7^{2r+1} + 2^{2r+1} &= 9M \qquad \text{where } M \in \mathbb{Z} \\ 7^{2r+1} &= 9M - 2^{2r+1} \end{aligned}$$

Consider when $k = r + 1$:

$$\begin{aligned} 7^{2(r+1)+1} + 2^{2(r+1)+1} &= 49\left(9M - 2^{2r+1}\right) + 4 \times 2^{2r+1} \quad \checkmark \quad \text{(by assumption)} \\ &= 9 \times 49M - 45 \times 2^{2r+1} \\ &= 9\left(49M - 5 \times 2^{2r+1}\right) \\ &= 9Q \qquad \text{where } Q \in \mathbb{Z} \text{ since } M, r \in \mathbb{Z} \end{aligned}$$

$\therefore$ Statement is true when $k = r + 1$ if it is true when $k = r$.

Hence by mathematical induction, the statement is true for $k \geq 1$. ✓

Question 12 (TBA)

(a) (3 marks)

✓ [1] for correctly applying t-formulas

✓ [1] for one correct solution of x

✓ [1] for both correct solutions of x

Using $t = \tan\dfrac{x}{2}$:

$$\begin{aligned} 2 - 3\sin x + 2\cos x &= 0 \\ 2 - 3\left(\frac{2t}{1+t^2}\right) + 2\left(\frac{1-t^2}{1+t^2}\right) &= 0 \qquad \checkmark \\ 2 + 2t^2 - 6t + 2 - 2t^2 &= 0 \\ t &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \tan\frac{x}{2} &= \frac{2}{3} \\ \frac{x}{2} &= 0.588... \\ x &= 1.18 \qquad \checkmark \qquad \text{(correct to two d.p.)} \end{aligned}$$

Check $x = \pi$:

$$\begin{aligned} \text{LHS} &= 2 - 3(0) + 2(-1) \\ &= 0 \\ \therefore \text{LHS} &= \text{RHS} \end{aligned}$$

Hence the solutions are $x = 1.18$ and $x = \pi$. ✓

(b) i. (1 mark)

✓ [1] for correct answer

By grouping the two together:

$$\begin{aligned}&(5-1)! \times 2 \qquad \checkmark\\ &= 48 \quad \text{ways}\end{aligned}$$

ii. (3 marks)

✓ [1] for correctly identifying all relevant cases

✓ [1] for correctly calculating one case

✓ [1] for correctly calculating both cases and correct final answer

Note: 3 people cannot sit next to each other, 5 people can sit freely.

Case 1: two of the three sit on the table of six

Consider that the number of ways to seat six people on a circular table is 5!:

$$\begin{aligned}&{}^3C_1 \times {}^5C_1 \times (5! - 48) \qquad \checkmark \qquad \text{(from i)}\\ &= 1080 \quad \text{ways}\end{aligned}$$

Case 2: all of the three sit on the table of six

$$\begin{aligned}&{}^5C_2 \times 2! \times 3! \qquad \checkmark\\ &= 120 \quad \text{ways}\end{aligned}$$

Hence 1200 ways. ✓

(c) i. (1 mark)

✓ [1] for correctly showing result

$$\overrightarrow{QR} = \frac{3}{2}\underset{\sim}{p}$$

$$\begin{aligned}\overrightarrow{OM} &= \frac{\mu}{\mu+1}\overrightarrow{OR}\\ \therefore \overrightarrow{OM} &= \frac{\mu}{\mu+1}\left(\underset{\sim}{q} + \frac{3}{2}\underset{\sim}{p}\right) \qquad \checkmark \qquad (1)\end{aligned}$$

ii. (1 mark)

✓ [1] for correctly showing result

$$\begin{aligned}\overrightarrow{OM} &= \underset{\sim}{p} + \overrightarrow{PM}\\ \therefore \overrightarrow{OM} &= \underset{\sim}{p} + \frac{\lambda}{\lambda+1}(\underset{\sim}{q} - \underset{\sim}{p}) \qquad \checkmark \qquad (2)\end{aligned}$$

iii. (3 marks)

- ✓ [1] for correct simultaneous equations for λ and μ
- ✓ [1] for one correct value of λ or μ
- ✓ [1] for correctly showing result

Equating (1) and (2) from parts i and ii:

$$\frac{3\mu}{2(\mu+1)}\underset{\sim}{p} + \frac{\mu}{\mu+1}\underset{\sim}{q} = \frac{1}{\lambda+1}\underset{\sim}{p} + \frac{\lambda}{\lambda+1}\underset{\sim}{q}$$

Since $\underset{\sim}{p}$ and $\underset{\sim}{q}$ are non-parallel vectors, equate coefficients:

$$\frac{3\mu}{2(\mu+1)} = \frac{1}{\lambda+1} \tag{2}$$

$$\frac{\mu}{\mu+1} = \frac{\lambda}{\lambda+1} \quad ✓ \tag{3}$$

Substitute (3) into (2):

$$\begin{aligned} \frac{3}{2}\left(\frac{\lambda}{\lambda+1}\right) &= \frac{1}{\lambda+1} \\ 3\lambda &= 2 \\ \therefore \lambda &= \frac{2}{3} \quad ✓ \end{aligned}$$

From (3):

$$\begin{aligned} \mu &= \lambda \\ \therefore \mu &= \frac{2}{3} \end{aligned}$$

Hence from (1),

$$\overrightarrow{OM} = \frac{3}{5}\underset{\sim}{p} + \frac{2}{5}\underset{\sim}{q} \quad ✓$$

Question 13 (TBA)

(a) i. (3 marks)

- ✓ [1] for correct equation for $\frac{c}{a}$ OR correct substitution from $P\left(-\frac{1}{2}\right)=0$
- ✓ [1] for correct equation for $-\frac{d}{a}$ OR correct substitution from $P'\left(-\frac{1}{2}\right)=0$
- ✓ [1] for correctly showing result

$$\frac{c}{a} = \left(-\frac{1}{2}\right)\gamma + \left(-\frac{1}{2}\right)\gamma + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)$$

$$\frac{c}{a} = \frac{1}{4} - \gamma \quad ✓ \qquad (1)$$

$$-\frac{d}{a} = \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\gamma$$

$$\frac{d}{a} = -\frac{1}{4}\gamma \quad ✓ \qquad (2)$$

From equations (1) & (2):

$$\begin{aligned} &4\left(\frac{c}{a}\right) - 16\left(\frac{d}{a}\right) \\ &= 4\left(\frac{1}{4} - \gamma\right) - 16\left(-\frac{1}{4}\gamma\right) \\ &= 1 - 4\gamma + 4\gamma \end{aligned}$$

$$\therefore 1 = 4\left(\frac{c}{a}\right) - 16\left(\frac{d}{a}\right) \quad ✓$$

Alternate solution:

$$\begin{aligned} P(x) &= ax^3 + bx^2 + cx + d \\ P'(x) &= 3ax^2 + 2bx + c \end{aligned}$$

$$P\left(-\frac{1}{2}\right) = 0$$

$$-\frac{1}{8}a + \frac{1}{4}b - \frac{1}{2}c + d = 0 \quad ✓$$

$$-a + 2b - 4c + 8d = 0 \qquad (1)$$

$$P'\left(-\frac{1}{2}\right) = 0$$

$$\frac{3}{4}a - b + c = 0 \quad ✓$$

$$3a - 4b + 4c = 0 \qquad (2)$$

Equations $2 \times (1) + (2)$:

$$a - 4c + 16d = 0$$

$$\therefore 1 = 4\left(\frac{c}{a}\right) - 16\left(\frac{d}{a}\right) \quad ✓$$

ii. (2 marks)

✓ [1] for significant progress

✓ [1] for correctly showing result

$$3c = 11d$$
$$3\left(\frac{c}{a}\right) = 11\left(\frac{d}{a}\right)$$

From part (i):

$$3\left(\frac{1}{4} - \gamma\right) = 11\left(-\frac{1}{4}\gamma\right) \quad ✓$$
$$\frac{3}{4} = \frac{\gamma}{4}$$

$$\therefore \gamma = 3 \quad ✓$$

Alternate solution:

$$3c = 11d$$
$$c = \frac{11}{3}d$$

Substitute c into part (i):

$$1 = 4\left(\frac{11}{3}\frac{d}{a}\right) - 16\left(\frac{d}{a}\right) \quad ✓$$
$$1 = -\frac{4}{3}\left(\frac{d}{a}\right)$$
$$\frac{3}{4} = -\frac{d}{a}$$
$$= \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)(\gamma)$$
$$\frac{3}{4} = \frac{\gamma}{4}$$

$$\therefore \gamma = 3 \quad ✓$$

Question 14 (4 marks)

- ✓ [1] for correct bounds for integrals in terms of θ
- ✓ [1] for correct substitution of integral
- ✓ [1] for correctly applying double angle
- ✓ [1] for correct final answer

$$\begin{aligned} x &= \cos\theta \\ \frac{dx}{d\theta} &= -\sin\theta \\ dx &= -\sin\theta\, d\theta \end{aligned}$$

$$\begin{aligned} \text{When } x = \frac{1}{2} \quad &, \quad \theta = \frac{\pi}{3} \\ \text{When } x = 0 \quad &, \quad \theta = \frac{\pi}{2} \qquad \checkmark \end{aligned}$$

$$\begin{aligned} \int_0^{\frac{1}{2}} \sqrt{1-x^2}\, dx &= \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \sqrt{1-\cos^2\theta} \cdot (-\sin\theta)\, d\theta \qquad \checkmark \\ &= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin^2\theta\, d\theta \\ &= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 2\theta)\, d\theta \qquad \checkmark \\ &= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\ &= \frac{1}{2}\left[\left(\frac{\pi}{2} - 0\right) - \left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)\right] \\ &= \frac{\pi}{12} + \frac{\sqrt{3}}{8} \qquad \checkmark \end{aligned}$$

Question 15 (4 marks)

- ✓ [1] for some progress in setting up expression for the volume
- ✓ [1] for significant progress in setting up expression for the volume
- ✓ [1] for correct primitive function
- ✓ [1] for correct final answer

$$y = 4x^3 - 2 \qquad\qquad y = \log_3 x + 2$$

$$x = \left(\frac{1}{4}(y+2)\right)^{\frac{1}{3}} \qquad\qquad x = 3^{y-2}$$

$$\begin{aligned}
V &= \left[\pi \int_0^2 \left(\left(\frac{1}{4}(y+2)\right)^{\frac{1}{3}}\right)^2 \, dy\right] - \left[\pi \int_0^2 \left(3^{y-2}\right)^2 \, dy\right] \\
&= \pi \int_0^2 4^{-\frac{2}{3}}(y+2)^{\frac{2}{3}} - 3^{2y-4} \, dy \\
&= \pi \left[4^{-\frac{2}{3}} \cdot \frac{3}{5}(y+2)^{\frac{5}{3}} - \frac{1}{2\ln 3} \cdot 3^{2y-4}\right]_0^2 \qquad \checkmark \\
&= \pi \left[\left(\frac{3}{5} \cdot \frac{4^{\frac{5}{3}}}{4^{\frac{2}{3}}}\right) - \frac{1}{\ln 9}\right] - \left[\left(\frac{3}{5} \cdot \frac{2^{\frac{5}{3}}}{2^{\frac{4}{3}}} - \frac{1}{81\ln 9}\right)\right] \\
&= \pi \left(\frac{12}{5} - \frac{3 \cdot \sqrt[3]{2}}{5} - \frac{80}{81 \ln 9}\right) \\
&= 3.75 \qquad \checkmark \qquad \text{(correct to two d.p.)}
\end{aligned}$$

Question 16 (4 marks)

- ✓ [1] for correct $\dfrac{dI}{dD^2}$ or equivalent progress
- ✓ [1] for one correct application of chain rule
- ✓ [1] for correct second application of chain rule
- ✓ [1] for correct final answer

$$\begin{aligned}\frac{dI}{dt} &= \frac{dI}{dD^2} \times \frac{dD^2}{dt} \\ &= \frac{dI}{dD^2} \times \left(\frac{dD^2}{dx} \times \frac{dx}{dt}\right) \qquad \checkmark \qquad (1)\end{aligned}$$

Given $P = 10$:

$$\begin{aligned}I &= \frac{5}{2\pi(D^2)} \\ \frac{dI}{dD^2} &= -\frac{5}{2\pi(D^2)^2} \qquad \checkmark\end{aligned}$$

$$\begin{aligned}D^2 &= x^2 + 9 \qquad (3) \\ \frac{dD^2}{dx} &= 2x\end{aligned}$$

When $D = 8$, from (3):

$$\begin{aligned}8^2 &= x^2 + 9 \\ x &= \sqrt{55}\end{aligned}$$

From (1), given $D = 8$ for which $x = \sqrt{55}$:

$$\begin{aligned}\frac{dI}{dt} &= -\frac{5}{2\pi \times 8^4} \times 2\sqrt{55} \times (-1.2) \qquad \checkmark \\ &= \frac{3\sqrt{55}}{2048\pi} \\ &= 0.003 \quad \text{W/m}^2 \qquad \checkmark \qquad \text{(correct to three d.p.)}\end{aligned}$$

Hence the rate of change of the sound intensity is 0.003 W/m^2.

Alternate solution:

$$\begin{aligned}\frac{dI}{dt} &= \frac{dI}{dD}\times\frac{dD}{dt}\\ &= \frac{dI}{dD}\times\left(\frac{dD}{dx}\times\frac{dx}{dt}\right) \qquad \checkmark \end{aligned} \tag{1}$$

Given $P = 10$:

$$\begin{aligned}I &= \frac{5}{2\pi D^2}\\ \frac{dI}{dD} &= -\frac{5}{\pi D^3} \qquad \checkmark\end{aligned}$$

$$\begin{aligned}D^2 &= x^2 + 9\\ D &= \sqrt{x^2+9} \qquad (3)\\ \frac{dD}{dx} &= -\frac{1}{2}\times\frac{2x}{\sqrt{x^2+9}}\\ \therefore \frac{dD}{dx} &= -\frac{x}{\sqrt{x^2+9}}\end{aligned}$$

When $D = 8$, from (3):

$$x = \sqrt{55}$$

From (1), given $D = 8$ for which $x = \sqrt{55}$:

$$\begin{aligned}\frac{dI}{dt} &= -\frac{5}{\pi\times 8^3}\times\frac{\sqrt{55}}{8}\times(-1.2) \qquad \checkmark\\ &= \frac{3\sqrt{55}}{2048\pi}\\ &= 0.003 \quad \mathrm{W/m^2} \qquad \checkmark \qquad \text{(correct to two d.p.)}\end{aligned}$$

Hence the rate of change of the sound intensity is 0.003 $\mathrm{W/m^2}$.

Question 17 (TBA)

(a) i. (3 marks)

✓ [1] for correct binomial expansion

✓ [1] for correct equation from independent term

✓ [1] for correctly showing final result with sufficient justification

$$\begin{aligned}\left(3x^a - \frac{1}{x^{4a}}\right)^{3n} &= \sum_{k=0}^{3n} \binom{3n}{k} (-1)^k \cdot 3^{3n-k} (x^a)^{3n-k} (x^{-4a})^k \quad \checkmark \\ &= \sum_{k=0}^{3n} \binom{3n}{k} (-1)^k \cdot 3^{3n-k} x^{3na-5ak}\end{aligned}$$

For the term independent of x:

$$\begin{aligned} 3na - 5ak &= 0 \quad \checkmark \\ a(3n-5k) &= 0 \\ 3n - 5k &= 0 \qquad (a \neq 0 \text{ since } a \in \mathbb{Z}^+) \\ n &= \frac{5k}{3}\end{aligned}$$

Since 3 and 5 have no common factors and $k, n \in \mathbb{Z}^+$, n must be a multiple of 5. ✓

Hence the binomial only has a term independent of x if n is a multiple of 5.

ii. (2 marks)

✓ [1] for correct value of k for independent term

✓ [1] for correct term

For $\left(3x^2 - \frac{1}{x^8}\right)^{75}$: $\quad a = 2,\ n = 25$

$$\left(3x^2 - \frac{1}{x^8}\right)^{75} = \sum_{k=0}^{75} \binom{75}{k} (-1)^k \cdot 3^{75-k} x^{150-10k}$$

Using part i:

$$\begin{aligned} 25 &= \frac{5k}{3} \\ k &= 15 \quad \checkmark \end{aligned}$$

Using $k = 15$, the term independent of x is:

$$T_{15} = -\binom{75}{15} \cdot 3^{60} \quad \checkmark$$

(b) i. (4 marks)

- ✓ [1] for correct integral by separating variables
- ✓ [1] for correct primitive function
- ✓ [1] for correctly applying initial conditions or equivalent progress
- ✓ [1] for correctly showing final result

$$\frac{dP}{dt} = \frac{1}{16\,000} rP(16\,000 - P)$$

Given that $P = 400$ when $t = 0$:

$$\begin{aligned}
\int_{400}^{P} \frac{1}{P(16\,000 - P)}\, dP &= \int_0^t \frac{1}{16\,000} r\, dt \qquad \checkmark \\
\frac{1}{16\,000}\int_{400}^{P} \frac{1}{P} + \frac{1}{16\,000 - P}\, dP &= \frac{1}{16\,000} r \int_0^t dt \qquad \text{(from given)} \\
\Big[\ln P - \ln(16\,000 - P)\Big]_{400}^{P} &= r\Big[t\Big]_0^t \qquad \checkmark \\
\ln\left(\frac{P}{16\,000 - P}\right) - \ln\frac{1}{39} &= rt \qquad \checkmark \\
\frac{39P}{16\,000 - P} &= e^{rt} \\
(39 + e^{rt})P &= 16\,000e^{rt} \\
P &= \frac{16\,000e^{rt}}{39 + e^{rt}}
\end{aligned}$$

$$\therefore \quad P = \frac{16\,000}{39e^{-rt} + 1} \qquad \checkmark$$

ii. (2 marks)

- ✓ [1] for applying given condition or equivalent merit
- ✓ [1] for correctly showing final result

When $t = 5,\ P = 4600$:

$$\begin{aligned}
4600 &= \frac{16\,000}{39e^{-5r} + 1} \qquad \checkmark \\
39e^{-5r} + 1 &= \frac{80}{23} \\
e^{-5r} &= \frac{19}{299}
\end{aligned}$$

$$\therefore r = -\frac{1}{5}\ln\left(\frac{19}{299}\right) \qquad \checkmark$$

iii. (2 marks)

✓ [1] for finding $P = 8000$ as population at which the growth rate is greatest

✓ [1] for correct final answer

At the **inflexion point** of the logistic curve, the tangent will have the greatest gradient and thus greatest population growth:

$$\frac{dP}{dt} = \frac{1}{16\,000}r(16\,000P - P^2)$$

$$\frac{d^2P}{dt^2} = \frac{1}{16\,000}r(16\,000 - 2P)$$

Since the logistic curve has an inflexion point, when $\dfrac{d^2P}{dt^2} = 0$:

$$P = 8000 \qquad \checkmark$$

$$8000 = \frac{16\,000}{39e^{-rt} + 1}$$

$$e^{-rt} = \frac{1}{39}$$

$$t = -\frac{1}{r}\ln\left(\frac{1}{39}\right)$$

$$= \frac{5\ln\left(\frac{1}{39}\right)}{\ln\left(\frac{19}{299}\right)} \qquad \text{(from ii.)}$$

$$\therefore \quad t = 6.64 \qquad \checkmark \qquad \text{(nearest two d.p.)}$$

∴ The population of mosquitoes is growing at its fastest rate, 6.64 years after they were released.

iv. (1 mark)

✓ [1] for correct answer with sufficient working

As $t \to \infty$, population approaches carrying capacity and population growth rate approaches zero.

Let $\dfrac{dP}{dt} = 0$:

$$\frac{1}{16\,000}rP(16\,000 - P) = 0$$

Since $P \neq 0$ as $t \to \infty$:

$$P = 16\,000 \qquad \checkmark$$

∴ The carrying capacity is 16 000 mosquitoes.

v. (2 marks)

✓ [1] for significant progress

✓ [1] for correct final answer with sufficient justification

Similarly to (iv), let $\dfrac{dP}{dt} = 0$, where $r = -\dfrac{1}{5}\ln\left(\dfrac{19}{299}\right)$ from (ii):

$$rP\left(1 - \frac{P}{16\,000}\right) - 0.09P = 0$$

$$P\left(r\left(1 - \frac{P}{16\,000}\right) - 0.09\right) = 0$$

$$1 - \frac{P}{16\,000} = \frac{0.09}{r} \qquad \checkmark \qquad \text{(since } P \neq 0 \text{ as } t \to \infty\text{)}$$

$$P = 16\,000\left(1 - \frac{0.09}{r}\right)$$

$$\therefore \quad P = 13\,388.53 \qquad \text{(correct to two d.p.)}$$

The new carrying capacity exceeds 13 000 mosquitoes.

Hence the researcher will **NOT** be able to protect the ecosystem from harm. ✓

Alternate solution:

Let $P = 13\,000$:

$$\frac{dP}{dt} = 173.55 \qquad \checkmark \qquad \text{(nearest two d.p.)}$$

Since $\dfrac{dP}{dt} > 0$, the population will continue to increase beyond 13 000.

Hence the researcher will **NOT** be able to protect the ecosystem from harm. ✓